On direct sums of countable, reduced, abelian p-groups

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In 1960, G. Kolettis [4] succeeded in generalizing Ulm's Theorem to arbitrary direct sums of countable, reduced, abelian p-groups. (The word 'countable' means finite or countably infinite.) However, his proof was rather complicated. Recently, P. Hill [1] simplified its proof. In § 1 of the present paper, we shall state a more natural proof of this theorem.

In the same paper [4], Kolettis gave an existence theorem for the same class of groups. The author [2] proved another type of existence theorem, which, according to our opinion, is clearer than Kolettis' one. But the proof, independent from Zippin's Theorem, was not easy. In § 2 of this paper, we shall state an easier proof, though using Zippin's Theorem.

§ 1. Isomorphism Theorem.

A subset P of the socle $G[p]$ of an abelian p-group G is called a *principal system* of G, when, for every ordinal α, the intersection $P \cap p^{\alpha}G$ forms a basis of $G[p] \cap p^{\alpha}G$. (Cf. the author's paper [3].) Every countable, reduced, abelian p-group contains a principal system.

Kolettis' Theorem. *Suppose that* $G=\sum_{\mu \in M} G_{\mu}$ *and* $H=\sum_{\nu \in N} H_{\nu}$ *are direct decompositions of abelian p-groups G and H into direct sums of countable, reduced, p-groups. If G and H have the same Ulm invariants, then they are isomorphic.*

PROOF. In the groups G_{μ} and H_{ν}, take principal systems P_{μ} of G_{μ} and Q_{ν} of H_{ν} for every $\mu \in M$, $\nu \in N$. The set-theoretic unions $P=\bigvee_{\mu \in M} P_{\mu}$ and $Q=\bigvee_{\nu \in N} Q_{\nu}$ are principal systems of G and H respectively. Since G and H have the same Ulm invariants, there exists a height-preserving one-to-one mapping ρ from P onto Q.

The family of sets $[Q_{\nu}]_{\nu \in N}$ is a classification of the set Q. The family of sets $[\rho P_{\mu}]_{\mu \in M}$ is also a classification of Q. As is well known, all the classifications of a given set form a lattice. We take the meet $[B_{\iota}]_{\iota \in I}$ of $[Q_{\nu}]_{\nu \in N}$ and $[\rho P_{\mu}]_{\mu \in M}$ in this lattice; two elements q and q' of Q are contained in the same B_{ι} if and only if there exists a system of elements

$$q=q_1,\ q_2,\ \cdots,\ q_r=q'$$

in Q, where, for every i $(1 \leqq i \leqq r-1)$, q_i and q_{i+1} are contained in the same Q_{ν_i} or in the same ρP_{μ_i}. Since every Q_ν and ρP_μ are countable, every B_ι is countable.

Put $\rho^{-1}B_\iota = A_\iota$, $\iota \in I$. The classification $[P_\mu]_{\mu \in M}$ is a refinement of the classification $[A_\iota]_{\iota \in I}$. Therefore, for every $\iota \in I$, there exists a subset M_ι of M such that $A_\iota = \bigvee_{\mu \in M_\iota} P_\mu$. Form the subgroup $\sum_{\mu \in M_\iota} G_\mu = \bar{A}_\iota$ in G. Then, A_ι is a principal system of $\bar{A}_\iota$. Similarly, $[Q_\nu]_{\nu \in N}$ is a refinement of $[B_\iota]_{\iota \in I}$ and there exists, for every $\iota \in I$, a subset N_ι of N such that $B_\iota = \bigvee_{\nu \in N_\iota} Q_\nu$. Form the subgroup $\sum_{\nu \in N_\iota} H_\nu = \bar{B}_\iota$. B_ι is a principal system of $\bar{B}_\iota$. The groups $\bar{A}_\iota$ and $\bar{B}_\iota$ are countable.

Since ρ induces a height-preserving one-to-one mapping from A_ι onto B_ι, for every $\iota \in I$, we have, by Ulm's Theorem, $\bar{A}_\iota \cong \bar{B}_\iota$, $\iota \in I$. Thus we have

$$G = \sum_{\iota \in I} \bar{A}_\iota \cong \sum_{\iota \in I} \bar{B}_\iota = H ,$$

as desired.

§ 2. Existence Theorem.

Let G be a reduced, abelian p-group of length λ. As is well known (cf. [4]), for every ordinal $\alpha < \lambda$, there exists a non-negative integer n such that the Ulm invariant $U_G(\alpha+n)$ is different from 0.

Let f be a cardinal-valued function on the ordinals. f is said to be of length λ when $f(\alpha)=0$ for every $\alpha \geqq \lambda$ and, moreover, λ is the least such ordinal. A function f of length λ is called admissible when, for every $\alpha < \lambda$, there exists a non-negative integer n such that $f(\alpha+n) \neq 0$. The Ulm invariants $U_G(\alpha)$ of any reduced p-group G of length λ is an admissible function of length λ.

A cardinal-valued function f on the ordinals is called countable when its length is a countable ordinal and its values are all countable cardinals. The Ulm invariants of any countable, reduced p-group is a countable function. We can formulate Zippin's Theorem as follows:

Zippin's Theorem. *Let f be a countable, admissible, cardinal-valued function of length λ on the ordinals. Then, there exists a countable, reduced, abelian p-group G of length λ such that $U_G(\alpha) = f(\alpha)$ for every ordinal α.*

Next, let f be a countable, admissible function. Then, for every ordinal α, we have

$$(1) \qquad \sum_{n<\omega} f(\alpha+n) \geqq \sum_{\beta \geqq \alpha+\omega} f(\beta) .$$

In fact, let λ be the length of f. When $\lambda \leqq \alpha+\omega$, the inequality (1) is obvious, since $f(\beta)=0$ $(\beta \geqq \lambda)$. When $\lambda > \alpha+\omega$, by the definition of admissible function, there exist infinitely many integers n_k such that $f(\alpha+n_k) \neq 0$. Thus the left-hand side of (1) is not smaller than $\aleph_0$. Since f is a countable function, the right-hand side of (1) is not greater than $\aleph_0$. Hence the inequality (1) holds.

Now, if a function g is a sum of countable, admissible functions, then we have, for every ordinal α,

$$\sum_{n<\omega} g(\alpha+n) \geqq \sum_{\beta \geqq \alpha+\omega} g(\beta) ,$$

and the length of g is not greater than Ω (the least non-countable ordinal). If G is a direct sum of countable, reduced p-groups, its Ulm invariants $U_G(\alpha)=g(\alpha)$ satisfy the above inequality, and the length of g is not greater than Ω.

Existence Theorem. *Let g be a cardinal-valued function on the ordinals. If g satisfies*

$$\sum_{n<\omega} g(\alpha+n) \geqq \sum_{\beta \geqq \alpha+\omega} g(\beta) , \tag{2}$$

for every ordinal α, and if the length λ of g is not greater than Ω, then there exists a p-group G of length λ, which is a direct sum of countable, reduced, abelian p-groups, satisfying $U_G(\alpha)=g(\alpha)$ for every ordinal α.

Proof. It we could express g as a sum of countable, admissible functions, then our theorem would follow immediately from Zippin's Theorem.

From the function g, we construct a family of sets $[A_\alpha]_\alpha$, indexed with all ordinals α's, such that, when $\alpha \neq \alpha'$, the intersection $A_\alpha \cap A_{\alpha'}$ is empty, and that, for every α, the power $|A_\alpha|$ is equal to $g(\alpha)$.

Let A be the set-theoretic union of all A_α's, i.e. $A=\bigvee_\alpha A_\alpha$. Using the condition (2), there exists, for every ordinal α, a one-to-one mapping ρ_α from the union $\bigvee_{\beta \geqq \alpha+\omega} A_\beta$ into the union $\bigvee_{n<\omega} A_{\alpha+n}$. Two elements x and y of A is called congruent modulo ρ_α when $\rho_\alpha x=y$ or $\rho_\alpha y=x$, and is denoted by $x \underset{\alpha}{\sim} y$. Two elements x and y of A is called congruent, and is denoted by $x \sim y$, when there exists a finite sequence of elements in A

$$x=x_1, x_2, \cdots, x_k=y ,$$

such that

$$x_1 \underset{\alpha_1}{\sim} x_2 \underset{\alpha_2}{\sim} x_3 \underset{\alpha_3}{\sim} \cdots \underset{\alpha_{k-1}}{\sim} x_k \tag{3}$$

for suitable ordinals $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$. By this notion of congruence, we can classify A, and let $[B_\gamma]_{\gamma\in\Gamma}$ be the family of these classes.

For every $\gamma\in\Gamma$, we define the function f_γ on the ordinals α's by $f_\gamma(\alpha)=|B_\gamma\cap A_\alpha|$. We shall prove that f_γ is a countable, admissible function.

First, every B_γ is countable. In fact, in (3), when we fix x_1, there exist at most countable x_2's. (Assume that $x_1\in A_\tau$. If $\rho_{\alpha_1}x_1=x_2$, then $\alpha_1+\omega\leqq\tau<\Omega$, and there exist at most countable such α_1's. If $\rho_{\alpha_1}x_2=x_1$, then there exists a finite ordinal n such that $\alpha_1+n=\tau$. For fixed τ, there exist at most finite such α_1's. Since ρ_{α_1} is one-to-one, there exists, for every α_1, at most one x_2 such that $\rho_{\alpha_1}x_2=x_1$.) Similarly, there exist at most countable x_3's. Continuing this process, we see that B_γ is (at most) countable. Therefore, the cardinal $f_\gamma(\alpha)=|B_\gamma\cap A_\alpha|$ is a countable one.

Let λ_γ be the length of f_γ, for every $\gamma\in\Gamma$. Since the length of g is not greater than Ω, we see that A_α is empty when $\alpha\geqq\Omega$. Hence we have $\lambda_\gamma\leqq\Omega$.

We shall prove that f_γ is an admissible function. Let α be any ordinal smaller than λ_γ. By the definition of length, there exists an ordinal α' $(\alpha\leqq\alpha'<\lambda_\gamma)$ such that $f_\gamma(\alpha')\neq0$. If $\alpha'=\alpha+n$ for a certain non-negative integer n, then it holds $f_\gamma(\alpha+n)\neq0$. If $\alpha'\geqq\alpha+\omega$, then, by the mapping ρ_α, an element x of $B_\gamma\cap A_{\alpha'}$, which is not empty by $f_\gamma(\alpha')\neq0$, is mapped to an element y of $A_{\alpha+n}$ for a certain non-negative integer n. Therefore, it holds $x\sim y$. Since x is an element of B_γ, we have $y\in B_\gamma$, and it follows $y\in B_\gamma\cap A_{\alpha+n}$. This shows $f_\gamma(\alpha+n)\neq0$. Thus, f_γ is an admissible function.

Finally, we shall prove that λ_γ is a countable ordinal. Assume that it is not so. Then we have $\lambda_\gamma=\Omega$. By the above admissibility, for every ordinal $\alpha<\Omega$, there exists a non-negative integer n such that $f_\gamma(\alpha+n)\neq0$. Therefore, for every limit ordinal $\alpha_0<\Omega$, the sum $\sum_{n<\omega}f_\gamma(\alpha_0+n)$ is equal to $\aleph_0$. Hence the sum $\sum_{0\leqq\alpha<\Omega}f_\gamma(\alpha)=|B_\gamma|$ is equal to the power $\aleph_1$ of all ordinals smaller than Ω. Since B_γ is countable, our result is a contradiction.

Now, every f_γ is a countable, admissible function, and for every ordinal α, we have

$$g(\alpha)=|A_\alpha|=\sum_{\gamma\in\Gamma}|B_\gamma\cap A_\alpha|=\sum_{\gamma\in\Gamma}f_\gamma(\alpha)\ .$$

Hence the function g is the sum of f_γ's, as desired.

Bibliography.

[1] P. Hill, *Sums of countable primary groups*, Proc. Amer. Math. Soc. **17** (1966), 1469–1470.

[2] K. Honda, *Realism in the theory of abelian groups*, II, this Commentarii **9** (1961), 11–28.

[3] K. Honda, *Realism in the theory of abelian groups*, III, this Commentarii **12** (1964), 75–111.

[4] G. Kolettis, *Direct sums of countable groups*, Duke Math. J. **27** (1960), 111–125.

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